THE UNIVERSITY OF ALBERTA

THE GIBBS PHENOMENON OF SEQUENCES AND FOURIER SERIES

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

by

ROBERT L. FORBES

EDMONTON, ALBERTA

INTRODUCTION

This thesis deals with a phenomenon arising in mathematical analysis called the Gibbs phenomenon. Much work has been done concerning this phenomenon and many of the known results are presented here. Some of the results, however, are new.

In chapter I we formulate a precise general definition of the phenomenon. Two other definitions occuring commonly in literature are discussed and shown to be equivalent to ours. A third definition, also occuring in literature, is shown to give rise to an inconsistency.

In chapter II we discuss limiting processes (summation methods), that is, methods by which generalized limits are found for sequences and functions.

The theory of the two preceding chapters is applied, in chapter III, to the Gibbs phenomenon as exhibted by the partial sums of the Fourier series

$\sum_{k=1}^{\infty} \frac{\sin k x}{k}$ and by the partial sums of the

Fourier series in general.

ACKNOWLEDGEMENTS

The writer would like to thank Dr. Lee Lorch for his guidance and advice.

TABLE OF CONTENTS

CHAPTER I

THE DEFINITION OF THE GIBBS PHENOMENON

Most authors define the Gibbs phenomenon as follows:

Definition 1.1: Let $\{f_n(x)\}$ be a sequence of function such that $\lim_{n \to \infty} f_n(x) = f(x)$ for $0 < |x - \xi| < h$. Then $\{f_n(x)\}$ is said to exhibit the Gibbs phenomenon at $x = \xi$ if

$$\overline{\lim_{\substack{n \to \infty \\ x \to \xi}}} f_n(x) > \overline{\lim_{x \to \xi}} f(x)$$

or

$$\underline{\lim_{\substack{n \to \infty \\ x \to \xi}}} f_n(x) < \underline{\lim_{x \to \xi}} f(x)$$

In order to discuss this definition it is necessary to know exactly what is meant by $\overline{\lim_{\substack{n \to \infty \\ x \to \xi}}} f_n(x)$, $\overline{\lim_{x \to \xi}} f(x)$, etc..

Definition 1.2 (Hobson [7]):

$$\overline{\lim_{\substack{n \to \infty \\ x \to \xi}}} f_n(x) = \lim_{n \to \infty} \sup \left[f_m(x);\ m > p_n ,\ 0 < |x - \xi| < \delta_n \right]$$

where

$$\lim_{n \to \infty} p_n = \infty,\ \delta_n > 0 \text{ for } n = 0, 1, \ldots, \text{ and } \lim_{n \to \infty} \delta_n = 0.$$

Hobson remarks and it is easily shown that $\overline{\lim_{\substack{n \to \infty \\ x \to \xi}}} f_n(x)$ is independent of the particular sequences $\{p_n\}$ and $\{\delta_n\}$

which we choose.

Definition 1.3 (Graves [4]):

$$\overline{\lim_{x \to \xi}}\ f(x) = \lim_{\delta \to 0}\ \sup\left[f(x)\ ;\ 0 < |x - \xi| < \delta \right].$$

Zalcwasser's [21] definition of the Gibbs phenomenon is very similar to Definition 1.1 except that $\overline{\lim}_{\substack{n \to \infty \\ x \to \xi}} f_n(x)$ is replaced by the quantity $\lim_{\delta \to 0} \left(\lim_{n \to \infty} \sup\left[f_m(x);\ m \geq n,\ 0 < |x - \xi| < \delta \right] \right)$.

Let us denote this quantity by

$$Z - \overline{\lim_{\substack{n \to \infty \\ x \to \xi}}}\ f_n(x)\ .$$

Theorem 1.1:
$$Z - \overline{\lim_{\substack{n \to \infty \\ x \to \xi}}}\ f_n(x) = \overline{\lim_{\substack{n \to \infty \\ x \to \xi}}}\ f_n(x)\ .$$

Proof: Let $\{\delta_n\}$ be a positive non-increasing null sequence. Let r be a positive integer. Then for $n \geq r$,

$$\sup\left[f_m(x)\ ;\ m \geq n\ ,\ 0 < |x - \xi| < \delta_n \right]$$

$$\leq \sup\left[f_m(x);\ m \geq n\ ,\ 0 < |x - \xi| < \delta_r \right]$$

Taking limits as $n \to \infty$ we obtain

$$\overline{\lim_{\substack{n \to \infty \\ x \to \xi}}}\ f_n(x) \leq \lim_{n \to \infty}\ \sup\left[f_m(x);\ m \geq n\ ,\ 0 < |x - \xi| < \delta_r \right]$$

which is true for every positive integer r. Taking limits as $r \to \infty$ we have

$$\overline{\lim_{\substack{n \to \infty \\ x \to \xi}}}\ f_n(x) \leq Z - \overline{\lim_{\substack{n \to \infty \\ x \to \xi}}}\ f_n(x)\ .$$

Also for $n \geqq r$,

$$\sup \left[f_m(x) \; ; \; m \geqq n, \; 0 < |x - \xi| < \delta_r \right]$$

$$\leqq \sup \left[f_m(x) \; ; \; m \geqq r, \; 0 < |x - \xi| < \delta_r \right] .$$

Again taking limits as $n \to \infty$ and $r \to \infty$ we obtain the results

$$Z - \overline{\lim_{\substack{n\to\infty \\ x \to \xi}}} \; f_n(x) \leqq \overline{\lim_{\substack{n\to\infty \\ x \to \xi}}} \; f_n(x) .$$

The following theorem is also of interest:

<u>Theorem 1.2:</u>

Let $Z' - \overline{\lim_{\substack{n\to\infty \\ x \to \xi}}} \; f_n(x) = \lim_{n \to \infty} \left(\lim_{\delta \to 0} \sup \left[f_m(x); \; m \geqq n, \; 0 < |x - \xi| < \delta \right] \right)$

Then $\quad Z' - \overline{\lim_{\substack{n\to\infty \\ x \to \xi}}} \; f_n(x) = \overline{\lim_{\substack{n\to\infty \\ x \to \xi}}} \; f_n(x) .$

The proof of this theorem is similar to that of Theorem 1.1.

Theorems 1.1 and 1.2 establish two alternative definitions of $\overline{\lim_{\substack{n\to\infty \\ x \to \xi}}} \; f_n(x)$ as well as the equivalence of Zalcwasser's definition and Definition 1.1.

<u>Theorem 1.3</u>: Under the same hypothesis as Definition 1.1,

$$\overline{\lim_{\substack{n\to\infty \\ x \to \xi}}} \; f_n(x) \geqq \overline{\lim_{x \to \xi}} \; f(x)$$

This theorem is necessary in order to justify Definition 1.1 and can be proved most easily using Zalcwasser's definition of $\overline{\lim_{\substack{n\to\infty \\ x \to \xi}}} \; f_n(x)$.

Proof: Choose $0 < \delta \leqq h$. Let $\varepsilon > 0$. Choose x_0 so that $0 < |x_0 - \xi| < \delta$ and also so that $f(x_0) > \sup\left[f(x);\ 0 < |x - \xi| < \delta\right] - \frac{\varepsilon}{2}$.

Let N be a positive integer such that $|f_n(x_0) - f(x_0)| < \frac{\varepsilon}{2}$ for $n \geqq N$. Then, for $n \geqq N$,

$$f_n(x_0) \leqq \sup\left[f_m(x);\ m \geqq n,\ 0 < |x - \xi| < \delta\right]$$

Hence for $n \geqq N$,

$$\sup\left[f_m(x);\ m \geqq n,\ 0 < |x - \xi| < \delta\right]$$
$$> \sup\left[f(x);\ 0 < |x - \xi| < \delta\right] - \varepsilon$$

Taking limits as $n \to \infty$ and $\delta \to 0$ we obtain the desired result.

We now turn our attention to what is generally known as the "Gibbs set" of the sequence $\{f_n(x)\}$ at the point $x = \xi$. We again assume the hypotheses of Definition 1.1.

Definition 1.4: Let $0 < \delta < h$. Let $X(\xi, \delta)$ be the set of all sequences $\{x_n\}$ such that $0 < |x_n - \xi| < \delta$ and $\lim_{n \to \infty} x_n = \xi$.

Let, for $\{x_n\} \in X(\xi, \delta)$, $G(\xi, x_n)$ be the set of the limit points of the sequence $\{f_n(x_n)\}$ (It is easily shown that $G(\xi, x_n)$ is independent of δ). Let $G(\xi) = \bigcup\left[G(\xi, x_n);\ \{x_n\} \in X(\xi, \delta)\right]$.

Then $G(\xi)$ is called the Gibbs set of the sequence $\{f_n(x)\}$ at $x = \xi$.

<u>Theorem 1.4:</u> $\sup G(\xi) = \overline{\lim_{\substack{n \to \infty \\ x \to \xi}}} f_n(x)$

<u>Proof</u>: Let $g \in G(\xi)$. Let $\{x_n\} \in X(\xi, \delta)$ such that g is a limit point of $\{f_n(x_n)\}$. Let $\{\delta_n\}$ be a null sequence such that $|x_n - \xi| < \delta_n$ for $n = 0, 1, \ldots$. (This is possible since $\lim_{n \to \infty} |x_n - \xi| = 0$).

Then $f_n(x_n) \leq \sup \left[f_m(x); m \geq n, 0 < |x - \xi| < \delta_n \right]$. It follows that $g \leq \overline{\lim_{\substack{m \to \infty \\ x \to \xi}}} f_n(x)$ and hence $\sup G(\xi) \leq \overline{\lim_{\substack{n \to \infty \\ x \to \xi}}} f_n(x)$.

Let $\{\delta_n\}$ be a positive null sequence. Let $\varepsilon > 0$. For $n = 0, 1, \ldots$, choose m_n and x_n such that $m_n \geq n$, $0 < |x_n - \xi| < \delta_n$ and also such that $f_{m_n}(x_n) > \sup \left[f_m(x); m \geq n, 0 < |x - \xi| < \delta_n \right] - \varepsilon$. Let g be a limit point of the sequence $\{f_{m_n}(x_n)\}$. It follows that $g \in G(\xi)$ and $g > \overline{\lim_{\substack{n \to \infty \\ x \to \xi}}} f_n(x) - \varepsilon$.

$$\therefore \sup G(\xi) \geq \overline{\lim_{\substack{n \to \infty \\ x \to \xi}}} f_n(x) .$$

Hence we arrive at a fourth definition of $\overline{\lim_{\substack{n \to \infty \\ x \to \xi}}} f_n(x)$ - one which will prove very useful when we come to discuss the Gibbs phenomenon as exhibited by the partial sums of Fourier series.

So far we have discussed only the Gibbs phenomenon as exhibited by a sequence $\{f_n(x)\}$ of functions.

Let us now consider a function $g_r(x)$ where r is a continuous real parameter which approaches the value ρ, $-\infty \leqq \rho \leqq \infty$, in some prescribed manner. In the following discussion it will be assumed that ρ is finite; trivial modifications are required in the case that ρ is infinite.

<u>Definition 1.5</u>: Let $g_r(x)$ exist for $0 < |r - \rho| < h$, and $0 < |x - \xi| < h_2$ and let $\lim\limits_{r \to \rho} g_r(x) = f(x)$ for $0 < |x - \xi| < h \leqq h_2$. Then $g_r(x)$ is said to exhibit the Gibbs phenomenon at $x = \xi$ if

$$\overline{\lim_{\substack{r \to \rho \\ x \to \xi}}}\ g_r(x) > \overline{\lim_{x \to \xi}}\ f(x)$$

or
$$\underline{\lim_{\substack{r \to \rho \\ x \to \xi}}}\ g_r(x) < \underline{\lim_{x \to \xi}}\ f(x) .$$

Here $\overline{\lim\limits_{\substack{r \to \rho \\ x \to \xi}}}\ g_r(x)$ is defined as follows:

<u>Definition 1.6 (Hobson [7])</u>:

$$\overline{\lim_{\substack{r \to \rho \\ x \to \xi}}}\ g_r(x) = \lim_{n \to \infty} \sup \left[g_r(x) ; 0 < |r - \rho| < \varepsilon_n, \ 0 < |x - \xi| < \delta_n \right]$$

where $\{\varepsilon_n\}$ and $\{\delta_n\}$ are positive null sequences.

It is easily shown that

$$\overline{\lim_{\substack{r \to \rho \\ x \to \xi}}}\ g_r(x)$$

$$= \lim_{\delta \to 0} \left(\lim_{\varepsilon \to 0} \sup \left[g_r(x) ; \ 0 < |r - \rho| < \varepsilon, \ 0 < |x - \xi| < \delta \right] \right)$$

$$= \lim_{\varepsilon \to 0} \left(\lim_{\delta \to 0} \sup \left[g_r(x) ; \ 0 < |r - \rho| < \varepsilon, \ 0 < |x - \xi| < \delta \right] \right) .$$

(The proofs are similar to the proof of Theorem 1.1).

In order to show the connection between Definition 1.5 and Definition 1.1 we let r approach ρ in the prescribed manner through a sequence $\{r_n\}$ and define $f_n(x) = g_{r_n}(x)$ for $n = 0, 1, \cdots$.

Then we have $\lim_{n \to \infty} f_n(x) = f(x)$ for $0 < |x - \xi| < h$,

<u>Theorem 1.5:</u> $$\overline{\lim_{\substack{n \to \infty \\ x \to \xi}}} f_n(x) \leqq \overline{\lim_{\substack{r \to \rho \\ x \to \xi}}} g_r(x)$$

when $\{f_n(x)\}$ and $g_r(x)$ are defined as above.

<u>Proof</u>: Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ be positive null sequences. Let, for $n = 0, 1, \cdots$, m_n be a positive integer such that $0 < |r_m - \rho| < \varepsilon_n$ for $m \geqq m_n$. Then, for $n = 0, 1, \cdots$,

$$\sup \left[g_r(x);\ 0 < |r - \rho| < \varepsilon_n,\ 0 < |x - \xi| < \delta_n \right]$$
$$\geqq \sup \left[g_{r_m}(x);\ m \geqq m_n\ ,\ 0 < |x - \xi| < \delta_n \right]$$
$$= \sup \left[f_m(x);\ m \geqq m_n\ ,\ 0 < |x - \xi| < \delta_n \right] .$$

Taking limits as $n \to \infty$ we obtain the desired result.

A corollary of this theorem is that if $\{f_n(x)\}$ exhibits the Gibbs phenomenon at $x = \xi$, then so does $g_r(x)$.

Zygmund [22] defines the Gibbs phenomenon as follows:

Suppose that a sequence $\{f_n(x)\}$ converges for $x_0 < x \leqq x_0 + h$ to a limit $f(x)$ and that $f(x_0 + 0)$ exists. Suppose that, when $n \to \infty$ and $x \to x_0$ independently, we have

$$\limsup f_n(x) > f(x_0 + 0)$$

or $$\liminf f_n(x) < f(x_0 + 0) ,$$

then we say that $\{f_n(x)\}$ shows the Gibbs phenomenon in the right - hand neighbourhood of x_0 . Similarly for the left - hand neighbourhood.

Here we shall interpret $\limsup f_n(x)$ as meaning

$$\overline{\lim_{\substack{n \to \infty \\ x \to x_0+}}} f_n(x) \quad \text{where}$$

$$\overline{\lim_{\substack{n \to \infty \\ x \to x_0+}}} f_n(x) = \limsup_{n \to \infty} \left[f_m(x);\ 0 < x - x_0 < \varepsilon_n;\ m \geq n \right]$$

where $\{\varepsilon_n\}$ is any positive null sequence.

Let us suppose that the Gibbs phenomenon does not occur. Then, by Zygmund's definition, we must have

$$\overline{\lim_{\substack{n \to \infty \\ x \to x_0+}}} f_n(x) = \underline{\lim_{\substack{n \to \infty \\ x \to x_0+}}} f_n(x) = f(x_0 + 0)$$

$$\therefore \quad \lim_{\substack{n \to \infty \\ x \to x_0+}} f_n(x) \quad \text{exists and equals} \quad f(x_0 + 0)$$

(See Hobson [7]).

Now let us suppose that the functions $f_n(x)$ are right - hand continuous at $x = x_0$ and that $\lim_{n \to \infty} f_n(x_0)$ exists and equals $f(x_0)$.

Then $f(x_0 + 0) = \lim_{\substack{n \to \infty \\ x \to x_0+}} f_n(x)$

$$= \lim_{n \to \infty} f_n(x_0)$$

$$= f(x_0)$$

and hence $f(x)$ is right - hand continuous at $x = x_0$.

Hence, by using Zygmund's definition, we have arrived at the following result:

If $f_n(x)$ is right - hand continuous at $x = x_0$ for $n = 0, 1, \cdots$ and if $\lim_{n \to \infty} f_n(x) = f(x)$ for $x_0 \leqq x \leqq x_0 + h$ and if $f(x)$ is discontinuous at $x = x_0$, then $\{f_n(x)\}$ exhibits the Gibbs phenomenon at $x = x_0$.

This result, however, contradicts the known result that the Cesàro means of order 1 of the sequence $\left\{ \sum_{k=1}^{n} \frac{\sin kx}{k} \right\}$ do not exhibit the Gibbs phenomenon. (See Chapter 3).

The difficulty is due to Zygmund's defining the Gibbs phenomenon at $x = x_0$ in the right and left - hand neighbourhoods of x_0 separately. The above analysis shows that this cannot be done in the way that he has tried to do it.

In our subsequent discussion we will adopt Definition 1.1 as our definition of the Gibbs phenomenon. We are, of course, at liberty to use any or all of the equivalent definitions of $\overline{\lim_{\substack{n \to \infty \\ x \to \xi}}} f_n(x)$.

CHAPTER II

LIMITING PROCESSES

The common limiting processes fall into two groups:

I. Let $A = (a_{nk})$ be a matrix of complex numbers. Let S be the set of all sequences $\{s_n\}$ of complex numbers such that $\sigma_n = \sum_{k=0}^{\infty} a_{nk} s_k$ exists for $n = 0, 1, \cdots$.

A is called a limiting process if there exists a sequence $\{s_n\} \in S$ such that

(i) $\lim_{n \to \infty} s_n$ does not exist.

(ii) $\lim_{n \to \infty} \sigma_n$ does exist.

If $\lim_{n \to \infty} \sigma_n$ exists and equals σ then we say that the sequence $\{s_n\}$ is A - limitable to the value σ and we write $A - \lim_{n \to \infty} s_n = \sigma$.

If, for any sequence $\{s_n\}$, $A - \lim_{n \to \infty} s_n$ exists and equals $\lim_{n \to \infty} s_n$ whenever the latter exists then A is called a regular limiting process. Necessary and sufficient conditions for the regularity of A are

(i) $\sum_{k=0}^{\infty} a_{nk} \leqq K$ for $n = 0, 1, \cdots$ where K is independent of n.

(ii) $\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 1$

(iii) $\lim_{n \to \infty} a_{nk} = 0$ for $k = 0, 1, \cdots$.

If $\{s_n\}$ is the sequence of partial sums of a series $\sum_{k=0}^{\infty} a_k$ and if $A - \lim_{n \to \infty} s_n$ exists and equals σ, then we say that the series $\sum_{k=0}^{\infty} a_k$ is A - summable to the value and we write $A - \sum_{k=0}^{\infty} a_k = \sigma$. For this reason a limiting process is also called a summation method.

II. Let $\underline{\emptyset} = \{\emptyset_n(Z)\}$ be a sequence of complex functions of a continuous complex parameter Z which approaches the value a in some prescribed manner. Let S' be the set of all sequences $\{s_n\}$ of complex numbers such that $\sigma(Z) = \sum_{k=0}^{\infty} \emptyset_k(Z)\, s_k$ exists for $|Z - a|$ small enough. Then $\underline{\emptyset}$ is called a limiting process if there exists a sequence $\{s_n\} \in S'$ such that

(i) $\lim_{n \to \infty} s_n$ does not exist.

(ii) $\lim_{Z \to a} \sigma(Z)$ does exist.

If $\lim_{Z \to a} \sigma(Z)$ exists and equals σ then we say that the the sequence $\{s_n\}$ is $\underline{\emptyset}$ - limitable to the value σ and we write $\underline{\emptyset} - \lim_{n \to \infty} s_n = \sigma$.

The actual difference between the two types of limiting processes is small when we consider that, in II, we may let Z approach a through a sequence $\{Z_n\}$ and let $A = (a_{nk}) = (\emptyset_k(Z_n))$ and $\sigma_n = \sigma(Z_n)$. If we do this, the regularity conditions for a limiting process $\underline{\emptyset}$ can be derived from those for a limiting process A. Hence necessary and sufficient conditions that $\underline{\emptyset}$ be a regular limiting process are

(i) $\sum_{k=0}^{\infty} \phi_k(Z) \leqq K$ for $|Z - a|$ small enough where K is independent of Z.

(ii) $\lim_{Z \to a} \sum_{k=0}^{\infty} \phi_k(Z) = 1$

(iii) $\lim_{Z \to a} \phi_k(Z) = 0$ for $k = 0, 1, \cdots$,

Examples of limiting processes of type I:

Cesàro's Method:

$$a_{nk} = \begin{cases} \dfrac{\binom{n-k+r-1}{r-1}}{\binom{n+r}{r}} & \text{for } k = 0, \cdots, n \\ 0 & \text{otherwise} \end{cases}$$

where $r > -1$.

Euler's Method:

$$a_{nk} = \begin{cases} \binom{n}{k} r^k (1-r)^{n-k} & \text{for } k = 0, \cdots, n \\ 0 & \text{otherwise} \end{cases}$$

where, for regularity, $0 < r \leqq 1$.

Hausdorff's Method:

$$a_{nk} = \begin{cases} \binom{n}{k} \int_0^1 r^k (1-r)^{n-k} \, d\psi(r) & \text{for } k = 0, \cdots, n \\ 0 & \text{otherwise} \end{cases}$$

where, for regularity, $\psi(r)$ is of bounded variation in $\langle 0,1 \rangle$, $\psi(1) - \psi(0) = 1$ and $\psi(0+) = \psi(0)$. The integral involved is a Stieltjes integral. $\psi(0)$ may be taken to be zero.

Nörlund's Method:

$$a_{nk} = \begin{cases} \dfrac{p_{n-k}}{\sum_{k=0}^{n} p_k} & \text{for } k = 0 \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

where $p_o > 0$, $p_n \geqq 0$ for $n = 1,2, \cdots$ and, for regularity,

$$\lim_{n\to\infty} \frac{p_n}{\sum_{k=0}^{n} p_k} = 0$$

<u>K - Methods</u>: To the regularity conditions we add the restriction

$$\sum_{k=1}^{\infty} k\,|a_{nk}| < \infty \qquad \text{for } n = 0,1,\cdots .$$

<u>Taylor's Method</u>: $\dfrac{(1-r)^{n+1}}{(1-r\theta)^{n+1}}\,\theta^n = \sum_{k=0}^{\infty} a_{nk}\,\theta^k$, $|r\theta| < 1$

ie.,

$$a_{nk} = \begin{cases} 0 & \text{for } k=0,1,\cdots, n-1 \\ (1-r)^{n+k} \binom{k}{n} r^{k-n} & \text{for } k=n,\cdots \end{cases}$$

<u>$[F,d_n]$ Method</u>: $a_{oo} = 0$

$$\prod_{j=1}^{n} \frac{\theta + d_j}{1 + d_j} = \sum_{k=0}^{\infty} a_{nk}\,\theta^k$$

where $d_n \geqq 0$ for $n=1,2,\cdots$ and, for regularity,

$$\sum_{n=1}^{\infty} \frac{1}{d_n+1} = \infty .$$

<u>Bernštein - Rogosinski Method</u>:

$$a_n = \cos^k \frac{\nu\pi}{2n+1} - \cos^k \frac{(\nu+1)\pi}{2n+1}$$

for $n, \nu = 0,1,\cdots$

Barlaz' Method:

$$a_{nk} = \begin{cases} e^{-x_n} \dfrac{x_n^{\,k}}{k!} & \text{for } k = 0, 1, \cdots \\ 0 & \text{otherwise} \end{cases}$$

where $\lim_{n \to \infty} x_n = \infty$ and, for regularity,

$$\lim_{n \to \infty} \frac{x_n - n}{\sqrt{n}} = -\infty .$$

Sonnenschein's Method: $$\left[f(Z) \right]^n = \sum_{k=0}^{\infty} a_{nk} Z^k \text{ for } n = 1, 2, \cdots$$

$$a_{oo} = 1 , \ a_{ok} = 0 \text{ for } k = 0, 1, \cdots .$$

where $f(Z)$ is analytic for $|Z| < R$, $R > 1$, $f(1) = 1$ and $|f(Z)| < 1$ for $|Z| \leq 1$, $Z \neq 1$. For regularity, $ReA \neq 0$ where A is defined by

$$f(Z) - Z^{f'(1)} = Ai^p (Z - 1)^p + O((Z - 1)^{p+1}) \text{ as } Z \to 1.$$

If $f(Z) \neq Z^k$ for some k then the above condition is also necessary for regularity.

Examples of limiting processes of type II:

Borel's Method: $$\phi_k (Z) = e^{-Z} \frac{Z^k}{k!}$$

where Z approaches ∞ through real values.

Abel's Method: $$\phi_k (Z) = (1 - Z) Z^k$$

where Z approaches $1 -$ through real values.

Riemann's Method: $\phi_{\nu}(Z) = \left(\frac{\sin \nu Z}{\nu Z}\right)^k - \left(\frac{\sin(\nu+1)Z}{(\nu+1)Z}\right)^k$

where Z approaches 0 through real values.

Riesz' Method: ((R, λ_n, α) method.):

$$\phi_k(\omega) = \begin{cases} \left(1 - \frac{\lambda_k}{\omega}\right)^{\alpha} - \left(1 - \frac{\lambda_{k+1}}{\omega}\right)^{\alpha} & \text{for } k=0,1,\cdots,m-1 \\ \left(1 - \frac{\lambda_k}{\omega}\right)^{\alpha} & \text{for } k=m \\ 0 & \text{otherwise} \end{cases}$$

where $0 = \lambda_0 < \lambda_1 < \lambda_2 \quad \cdots$, m is the largest integer k such that $\lambda_k < \omega$ and ω approaches ∞ through real values.

CHAPTER III

THE GIBBS PHENOMENON AS DISPLAYED BY THE PARTIAL SUMS OF FOURIER SERIES

The Classical Example of the Gibbs phenomenon:

Let f be the periodic function of period 2π such that

$$f(x) = \begin{cases} 0 & \text{for } x = 0 \\ \dfrac{\pi - x}{2} & \text{for } 0 < x < 2\pi \end{cases}$$

Then $\overline{\lim\limits_{x \to 0}}\ f(x) = \dfrac{\pi}{2}$.

$f(x)$ is represented by the Fourier series $\sum\limits_{\nu=1}^{\infty} \dfrac{\sin \nu x}{\nu}$,

ie., $$\lim_{n \to \infty} s_n(x) = \lim_{n \to \infty} \sum_{\nu=1}^{n} \frac{\sin \nu x}{\nu} = f(x) .$$

$$s_n(x) = \int_0^x \sum_{\nu=1}^{n} \cos \nu u \ du$$

$$= -\frac{x}{2} + \int_0^x \frac{\sin\,(n+\frac{1}{2})u}{2 \sin \frac{u}{2}} \ du$$

Let $\{x_n\}$ be a sequence such that $\lim\limits_{n \to \infty} x_n = 0+$. Then it is easily shown that $s_n(x_n) = \int_0^{nx_n} \dfrac{\sin u}{u} \ du + o(1)$ as $n \to \infty$. Now the function $\int_0^x \dfrac{\sin u}{u} \ du \ (x \geq 0)$ attains its maximum value at $x = \pi$.

$$\therefore \quad \overline{\lim_{\substack{n \to \infty \\ x \to 0}}} \; s_n(x) \;\leqq\; \int_0^{\pi} \frac{\sin u}{u}\, du$$

However by putting $\{x_n\} = \frac{\pi}{n}$ we see that $\int_0^{\pi} \frac{\sin u}{u}\, du$ belongs to the Gibbs set of $\{s_n(x)\}$ at $x = 0$.

$$\therefore \quad \overline{\lim_{\substack{n \to \infty \\ x \to 0}}} \; s_n(x) = \int_0^{\pi} \frac{\sin u}{u}\, du$$

$$= (1.1789797\cdots)\,\frac{\pi}{2}$$

$$> \frac{\pi}{2}$$

$$= \overline{\lim_{x \to 0}} \; f(x).$$

Hence the Gibbs phenomenon occurs at $x = 0$.

If we now transform the sequence $\{s_n(x)\}$ by means of a regular limiting process, the derived sequence will also converge to $f(x)$. The question arises as to whether this sequence also exhibits the Gibbs at $x = 0$. The following results are known:

<u>Theorem 3.1</u>: Let $A = (a_{nk})$ be a regular limiting process. Let a_{nk} be real and $\geqq 0$ for $n,k = 0, 1, \cdots$. Let $\sigma_n(x) = \sum_{k=0}^{\infty} a_{nk}\, s_k(x)$. Then $\overline{\lim_{\substack{n \to \infty \\ x \to 0}}}\, \sigma_n(x) \leqq \overline{\lim_{\substack{n \to \infty \\ x \to 0}}}\, s_n(x)$. (In fact, this theorem is true for any sequence $\{s_n(x)\}$ of real functions with 0 replaced by ξ).

This theorem is well - known and is stated as an exercise in Rogosinski [17] . For lack of a reference we give the following proof:

<u>Proof</u>: Let g be a member of the Gibbs set of $\{\sigma_n(x)\}$ at $x = 0$. Then g is the limit point of a sequence $\{\sigma_n(x_n)\}$ where $\{x_n\}$ is some null sequence. Let $\{m_n\}$ be a non-decreasing subsequence of $\{n\}$ such that $\lim_{n \to \infty} \sigma_{m_n}(x_{m_n}) = g$.

Now $\sigma_{m_n}(x_{m_n}) = \sum_{k=0}^{\infty} a_{m_n k}\, s_k(x_{m_n})$.

Let $\{\delta_n\}$ be a null sequence such that $|x_{m_n}| < \delta_n$ for $n = 0, 1, \cdots$ and let r be any positive integer. Since $a_{nk} \geqq 0$ for $n, k = 0, 1, \cdots$, it follows that

$$\sigma_{m_n}(x_{m_n}) = \sum_{k=0}^{r-1} a_{m_n k}\, s_k(x_{m_n}) + \sup\left[s_k(x);\ k \geqq r,\ 0<|x|<\delta_n\right] \cdot \sum_{k=r}^{\infty} a_{m_n k}$$

As $n \to \infty$, $\sum_{k=0}^{r-1} a_{m_n k}\, s_k(x_{m_n}) \to 0$ and $\sum_{k=r}^{\infty} a_{m_n k} \to 1$ by the regularity conditions.

$$\therefore \quad g \leqq \limsup_{n \to \infty} \left[s_k(x);\ k \geqq r,\ 0 < |x| < \delta_n\right]$$

and this is true for every positive integer r.

$$\therefore \quad g \leqq \lim_{r\to\infty} (\limsup_{n\to\infty} \left[s_k(x);\ k \geqq r,\ 0 < |x| < \delta_n \right]$$

$$= \overline{\lim_{\substack{n\to\infty\\ x\to 0}}}\ s_n(x) \quad \text{by definition.}$$

g was any element of the Gibbs set of $\{\sigma_n(x)\}$ at $x = 0$.

$$\therefore \quad \overline{\lim_{\substack{n\to\infty\\ x\to 0}}}\ \sigma_n(x) \leqq \overline{\lim_{\substack{n\to\infty\\ x\to 0}}}\ s_n(x) .$$

Theorem 3.2: Let $\underline{\Phi} = \{\phi_k(Z)\}$ be a regular limiting process where Z approaches a real number a through real values. Let $\phi_k(Z)$ be real and $\geqq 0$ for all k and $Z - a$ sufficiently small. Let $\sigma_Z(t) = \sum_{k=0}^{\infty} \phi_k(Z)\ s_k(t)$. Then

$$\overline{\lim_{\substack{Z\to a\\ t\to 0}}}\ {}_Z(t) \leqq \overline{\lim_{\substack{n\to\infty\\ t\to 0}}}\ s_n(t) .$$

The proof of this theorem is similar to the proof of Theorem 3.1.

Theorem 3.3 (Cramér [3]): Let $\sigma_n^r(x)$ be the n th Cesaro mean of order r of $\{s_n(x)\}$. Then there exists an absolute constant r_o, $0 < r_o < 1$, such that $\{\sigma_n^r(x)\}$ exhibits the Gibbs phenomenon at $x = 0$ for $r < r_o$ but not for $r \geqq r_o$.

Theorem 3.4 (Szasz [19]): Let $\sigma_n^r(x)$ be the nth Euler mean of order r of $\{s_n(x)\}$ where $0 < r \leqq 1$. Let $\{x_n\}$ be a sequence such that $\lim_{n\to\infty} x_n = 0+$ and $\lim_{n\to\infty} n\,x_n^2 = 0$.

Then $\sigma_n^r(x_n) = \int_o^{rnx_n} \frac{\sin u}{u}\,du + o(1)$ as $n \to \infty$.

Corollary: Because of Theorem 3.1, $\overline{\lim_{\substack{n \to \infty \\ x \to 0}}} \sigma_n^r(x) = \int_0^{\pi} \frac{\sin u}{u} du$

and the Gibbs phenomenon occurs at $x = 0$.

Theorem 3.5 (Lorch [13]): Let $B_x(t)$ be the x th Borel mean of $\{s_n(t)\}$. Let $\lim_{x \to \infty} t_x = 0+$. Then

$$B_x(t_x) = \int_0^{xt_x} \frac{\sin u}{u} du + o(1) \quad \text{as} \quad x \to \infty .$$

Corollary: Because of Theorem 3.2,

$$\overline{\lim_{\substack{x \to \infty \\ t \to 0+}}} B_x(t) = \int_0^{\pi} \frac{\sin u}{u} du$$ and the Gibbs

phenomenon occurs at $t = 0$.

Theorem 3.6: Let $\sigma_n^r(t)$ be the n th Taylor mean of order r of $\{s_n(t)\}$ where $0 \leq r < 1$. Let $\lim_{n \to \infty} t_n = 0+$ and $\lim_{n \to \infty} n\, t_n^4 = 0$. Then

$$\sigma_n^r(t_n) = \int_0^{\frac{nt_n}{1-r}} \frac{\sin u}{u} du + o(1) \quad \text{as} \quad n \to \infty .$$

This theorem is a modification of Miracle's [14]. The first lemma, however, is original.

Proof: $$s_n(t) = \int_0^t \frac{\sin nu}{u} du + O(t) \quad \text{as} \quad t \to 0$$

$$\therefore \quad \sigma_n^r(t) = \int_0^t \sum_{k=0}^{\infty} a_{nk} \sin ku \frac{du}{u} + O(t) \quad \text{as} \quad t \to 0$$

$$= \int_0^t \mathrm{Im}\left(\sum_{k=0}^{\infty} a_{nk} \; e^{iku} \right) \frac{du}{u} + O(t) \quad \text{as} \quad t \to 0.$$

$$= \int_0^t \mathrm{Im} \left(\frac{(1-r)^{n+1} \; e^{inu}}{(1 - r\, e^{iu})^{n+1}} \right) \frac{du}{u} + O(t) \quad \text{as} \quad t \to 0$$

$$= \int_0^t \left(\frac{1-r}{\rho}\right)^{n+1} \frac{\sin\,(nu + (n+1)\theta)}{u} \, du + O(t) \quad \text{as } t \to 0$$

where $\rho e^{-i\theta} = 1 - r\,e^{iu}$

$$\rho\cos\theta = 1 - r\cos u.$$

$$\rho\sin\theta = r\sin u$$

$$\rho^2 = 1 - 2r\cos u + r^2$$

$$= (1-r)^2 + 4r\sin^2\frac{u}{2}$$

$$= (1+r)^2 - 4r\cos^2\frac{u}{2}$$

$$\therefore \quad 0 < 1 - r \leqq \rho \leqq 1 + r .$$

<u>Lemma</u>: $$\left| \left(\frac{1-r}{\rho}\right)^n - e^{-\frac{nru^2}{2(1-r)^2}} \right| \leqq A n u^4$$

where A is a constant.

<u>Proof</u>: $$\left| \frac{u^2}{(1-r)^2} - \frac{4\sin^2\frac{u}{2}}{\rho^2} \right|$$

$$= \frac{\rho^2\left(u^2 - 4\sin^2\frac{u}{2}\right) + 4\sin^2\frac{u}{2}\left(\rho^2 - (1-r)^2\right)}{(1-r)^2\,\rho^2}$$

$$= \frac{4\rho^2 \left(\frac{u}{2} + \sin\frac{u}{2}\right)\left(\frac{u}{2} - \sin\frac{u}{2}\right) + 16r\sin^4\frac{u}{2}}{(1-r)^2\ \rho^2}$$

$$\leqq \frac{4\,(1+r)^2\left(\frac{u}{2} + \frac{u}{2}\right)\frac{1}{3!}\left(\frac{u}{2}\right)^3 + 16r\left(\frac{u}{2}\right)^4}{(1-r)^4}$$

$= Bu^4$ where B is a constant.

Also,
$$\left| e^{\frac{-ru^2}{(1-r)^2}} - \left(1 - \frac{ru^2}{(1-r)^2}\right)\right| \leqq \frac{r^2 u^4}{2(1-r)^4} = Cu^4$$

where C is a constant.

It follows that

$$\left| \left(\frac{1-r}{\rho}\right)^2 - e^{\frac{-ru^2}{(1-r)^2}} \right| \leqq (rB + C)\, u^4$$

For convenience, let $a = \left(\frac{1-r}{\rho}\right)^2$, $b = e^{\frac{-ru^2}{(1-r)^2}}$ and $f(x) = x^{\frac{n}{2}}$.

By the mean value theorem we have

$$\frac{f(a) - f(b)}{a - b} = f'(\xi) \text{ where } \min[a,b] \leqq \xi \leqq \max[a,b] .$$

Hence $0 < \xi \leqq 1$ and $\left|f(a) - f(b)\right| \leqq \frac{n}{2}\,|a - b|$

$$\therefore \left| \left(\frac{1-r}{\rho}\right)^n - e^{\frac{-nru^2}{2(1-r)^2}} \right| \leqq \frac{n}{2}(rB + C)\, u^4 = A_n u^4$$

where A is a constant.

This proves the lemma.

Lemma: (Miracle [14])

$$\left| \theta - \frac{ru}{1-r} \right| \leqq Au^3 \quad \text{where } A \text{ is a constant.}$$

Using these results and the assumption that $\lim_{n \to \infty} n\, t_n^{\,4} = 0$ it is easily shown that

$$\sigma_n^r(t_n) = \int_0^{t_n} e^{-\frac{nru^2}{2(1-r)^2}} \frac{\sin(\frac{nu}{1-r})}{u} \, du + o(1)$$

as $n \to \infty$.

Following the example of Lorch's paper [13] it is easily shown that the exponential may be replaced by 1 with error $o(1)$ as $n \to \infty$.

$$\therefore \sigma_n^r(t_n) = \int_0^{\frac{nt_n}{1-r}} \frac{\sin u}{u} \, du + o(1) \quad \text{as } n \to \infty .$$

Corollary: Because of Theorem 3.1,

$$\overline{\lim_{\substack{n \to \infty \\ t \to 0}}} \sigma_n^r(t) = \int_0^{\pi} \frac{\sin u}{u} \, du \quad \text{and the Gibbs}$$

phenomenon occurs at $t = 0$.

Remarks:

(1). In the Szasz theorem [19] on Euler means the restriction $\lim_{n \to \infty} nx_n^{\,2} = 0$ on the sequence $\{x_n\}$ may be replaced by $\lim_{n \to \infty} nx_n^{\,4} = 0$. The proof requires computations

similar to those above.

(2). Using the lemmas of the preceding theorem, the Lebesgue constants for Taylor summability are easily computed. It can be shown that

$$L_T^r(n) = L_B\left(\frac{n}{r}\right) + o(1) \quad \text{as} \quad n \to \infty$$

where $L_T^r(n)$ is the nth Taylor - Lebesgue constant and $L_B(x)$ is the xth Borel - Lebesgue constant.

Theorem 3.7 (Modification of Miracle's [14]):

Let $\sigma_n(x)$ be the nth $[F,d_n]$ mean of $\{s_n(x)\}$. Let $\{x_n\}$ be a sequence such that $\lim\limits_{n\to\infty} x_n = 0+$ and $\lim\limits_{n\to\infty} H_n x_n^3 = 0$ where

$$H_n = \sum_{j=1}^{n} \frac{1}{1+d_j}$$

Then $$\sigma_n(x_n) = \int_0^{H_n x_n} \frac{\sin u}{u}\, du + o(1) \quad \text{as} \quad n \to \infty .$$

Proof: This theorem is easily proved using Miracle's computations.

Corollary: Because of Theorem 3.1,

$$\overline{\lim_{\substack{n\to\infty \\ x\to 0+}}}\ \sigma_n(x) = \int_0^{\pi} \frac{\sin u}{u}\, du$$ and the Gibbs phenomenon occurs at $x = 0$.

Theorem 3.8 (Harsiladge [16]):

Let $B_n^k(x)$ be the nth Berštein - Rogosinski mean of order k of $\{s_n(x)\}$. Then $\{B_n^k(x)\}$ exhibits the Gibbs phenomenon at $x = 0$ for $k = 1, 2, \cdots$ although in rapidly decreasing measure since

$$\overline{\lim_{\substack{n \to \infty \\ x \to 0}}} B_n^k(x) - \frac{\pi}{2} < \frac{1}{2^k(k+1)}$$

Theorem 3.9 (Lee [12]):

Let $\sigma_x(t)$ be the xth Riemann mean of order 1 of $\{s_n(x)\}$. Then $\{\sigma_x(t)\}$ does not exhibit the Gibbs phenomenon at $t = 0$.

Theorem 3.10 (Cheng [2]):

Let $\sigma_\omega^\alpha(t)$ be the th (R, n^2, α) mean of $\{s_n(t)\}$. Let $\alpha > 0$. Then

$$\overline{\lim_{\substack{\omega \to \infty \\ t \to 0}}} \sigma_\omega^\alpha(t) = \frac{\pi}{2} \quad 2^\alpha \Gamma(\alpha + 1) \int_0^\gamma \frac{J_{\frac{1}{2}+\alpha}(u)}{u} \, du > \frac{\pi}{2}$$

where γ is the smallest positive zero of $J_{\frac{1}{2}+\alpha}(u)$.

Corollary: The Gibbs phenomenon occurs at $t = 0$.

Theorem 3.11 (Sledd [18]):

Let $\sigma_n(t)$ be the nth regular Sonnenschien mean of $\{s_n(t)\}$. Then $\{\sigma_n(t)\}$ exhibits the Gibbs phenomenon at $t = 0$.

Theorem 3.12 (Szasz [20]):

Let $h_n(t)$ be the nth regular Hausdorff mean with weight function ψ of $\{s_n(t)\}$.
Then

$$h_n(t_n) = \int_0^1 \int_0^{nt_n} \frac{\sin ry}{y}\, dy\, d\psi(r) + o(1) \quad \text{as} \quad n \to \infty$$

$$= \int_0^1 (1 - \psi(r)) \frac{\sin nt_n r}{r}\, dr + o(1) \quad \text{as} \quad n \to \infty .$$

Theorem 3.13 (Newman [15]):

Let $h_n(t)$ be the nth regular Hausdorff mean with weight function g of $\{s_n(t)\}$. Then if g is discontinuous and

$$\int_0^1 \frac{|d\, g(r)|}{r^2} < \infty , \quad \{h_n(t)\} \text{ exhibits the Gibbs}$$

phenomenon at $t = 0$.

<u>Theorem 3.14 (Kuttner 11)</u>:

Let $\sigma_\omega^\kappa(x)$ be the ωth (R, n^λ, k) mean of $\{s_n(x)\}$. Then, if $0 < \lambda < 2$, there is a function $r(\lambda)$ such that $\{\sigma_\omega^\kappa(x)\}$ exhibits the Gibbs phenomenon at $x = 0$ for $k < r(\lambda)$ but not for $k \geqq r(\lambda)$. The function $r(\lambda)$ is continuous and increasing, $\lim_{\lambda \to 0} r(\lambda) = 0$, $r(1) = r_0$, Cramér's constant and $\lim_{\lambda \to 2} r(\lambda) = 0$. If $\lambda \geqq 2$, $\{\sigma_\omega^\kappa(x)\}$ exhibits the Gibbs phenomenon at $x = 0$ for all k.

<u>Theorem 3.15</u>:

Let $B(x_n,t)$ be the nth Barlaz mean of $\{s_n(t)\}$ and let $\{x_n\}$ be chosen so that the method is regular. Let $\{t_n\}$ be a sequence such that $\lim_{n \to \infty} t_n = 0+$ and $\lim_{n \to \infty} x_n t_n^3$.

$$\text{Then} \quad B(x_n,t_n) = \int_0^{x_n t_n} \frac{\sin u}{u}\, du + o(1) \quad \text{as} \quad n \to \infty .$$

This is an original result due to the author.

<u>Proof</u>:

$$B(x_n,t_n) = e^{-x_n} \sum_{\nu=0}^{n} \left\{ -\frac{t_n}{2} + \int_0^{t_n} \frac{\sin(\nu+\frac{1}{2})u}{2\sin\frac{u}{2}}\, du \right\} \frac{x_n^\nu}{\nu!}$$

(see Barlaz [1])

$$= e^{-x_n} \int_0^{t_n} \sum_{\nu=0}^{n} \sin(\nu+\tfrac{1}{2})u \; \frac{x_n^\nu}{\nu!} \frac{du}{2\sin\frac{u}{2}} + o(1) \quad \text{as} \quad n \to \infty .$$

$$\sum_{\nu=0}^{n} \sin(\nu+\tfrac{1}{2})u \; \frac{x_n^{\nu}}{\nu!}$$

$$= \operatorname{Im}\left\{ \sum_{\nu=0}^{n} e^{i(\nu+\frac{1}{2})u} \frac{x_n^{\nu}}{\nu!} \right\}$$

$$= \operatorname{Im}\left\{ e^{\frac{iu}{2}} \sum_{\nu=0}^{n} \frac{(x_n e^{iu})}{\nu!} \right\}$$

$$= \operatorname{Im}\left\{ e^{\frac{iu}{2}} \left(e^{x_n e^{iu}} - \frac{e^{x_n e^{iu}}}{n!} \int_0^{x_n e^{iu}} Z^n e^{-Z}\, dZ \right) \right\}$$

$$= e^{x_n \cos u} \sin\left(x_n \sin u + \frac{u}{2}\right)$$

$$- \operatorname{Im}\left\{ \frac{1}{n!} \int_0^{x_n} \omega^n e^{(x_n-\omega)e^{iu} + (n+\frac{3}{2})iu}\, d\omega \right\}$$

$$(Z = \omega e^{iu})$$

$$\therefore\; B(x_n, t_n) = \int_0^{t_n} e^{-(1-\cos u)x_n} \frac{\sin(x_n \sin u + \frac{u}{2})}{2\sin\frac{u}{2}}\, du$$

$$- D(x_n, t_n) + o(1) \quad \text{as} \quad n \to \infty .$$

$$= \int_0^{x_n t_n} \frac{\sin u}{u}\, du - D(x_n, t_n) + o(1) \quad \text{as} \quad n \to \infty$$

by Lorch's Theorem [13] .

$$D(x_n,t_n) = \frac{e^{-x_n}}{n!} \int_0^{t_n} \int_0^{x_n} \omega^n e^{-(x_n-\omega)\cos u} \sin((x_n-\omega)\sin u + (n+\tfrac{3}{2})u)\,d\omega \cdot \frac{du}{2\sin\frac{u}{2}}$$

$$= \frac{1}{n!} \int_0^{t_n} \int_0^{x_n} \omega^n e^{-\omega} e^{-(x_n-\omega)2\sin^2\frac{u}{2}} \sin((x_n-\omega)\sin u + (n+\tfrac{3}{2})u)\,d\omega \cdot \frac{du}{2\sin\frac{u}{2}}$$

$$= \frac{A}{n!} \int_0^{t_n} \int_0^{x_n} \omega^n e^{-\omega} \sin((x_n-\omega)\sin u + (n+\tfrac{3}{2})u)\,d\omega \quad \frac{du}{2\sin\frac{u}{2}}$$

where $0 < A \leq 1$.

Now $$\frac{1}{n!} \int_0^{x_n} \omega^n e^{-\omega} d\omega = 1 - e^{-x_n} \sum_{\nu=0}^{n} \frac{x_n^\nu}{\nu!}$$

For regularity, $$\lim_{n\to\infty} e^{-x_n} \sum_{\nu=0}^{n} \frac{x_n^\nu}{\nu!} = 1$$

$$\therefore \quad \frac{1}{n!} \int_0^{x_n} \omega^n e^{-\omega} d\omega = o(1) \quad \text{as} \quad n \to \infty.$$

Using this fact and the fact that $u - \sin u \leq \frac{u^3}{6}$ for $u \geq 0$ and $\sin u \leq \frac{2u}{\pi}$ for $0 \leq u \leq \frac{\pi}{2}$, it is easily shown that

$$D(x_n,t_n) = \frac{A}{n!}\int_0^{t_n}\int_0^{x_n} \omega^n e^{-\omega}\sin(x_n-\omega+n)u \; d\omega \, \frac{du}{u} + o(1) \quad \text{as } n\to\infty .$$

$$= \frac{A}{n!}\int_0^{t_n}\int_0^{x_n} f(\omega,u) \; d\omega\, du + o(1) \quad \text{as } n\to\infty .$$

$\int_0^{t_n} f(\omega,u)\,du$ exists for all ω and $\int_0^{x_n} f(\omega,u)\,d\omega$ exists for all u. Hence we can interchange the order of integration.

$$D(x_n,t_n) = \frac{A}{n!}\int_0^{x_n} \omega^n e^{-\omega}\int_0^{t_n} \frac{\sin(x_n-\omega+n)u}{u}\, du\, d\omega + o(1) \quad \text{as } n\to\infty .$$

$$\therefore \left|D(x_n,t_n)\right| \leq \frac{A}{n!}\int_0^{x_n} \omega^n e^{-\omega}\left|\int_0^{(x_n-\omega+n)t_n} \frac{\sin u}{u}\, du\right| d\omega + o(1) \quad \text{as } n\to\infty .$$

$$\leq A\int_0^{\pi} \frac{\sin u}{u}\, du \; \frac{1}{n!}\int_0^{x_n} \omega^n e^{-\omega}\, d\omega + o(1) \quad \text{as } n\to\infty .$$

$$= o(1) \quad \text{as } n\to\infty .$$

This proves the theorem.

Corollary:

$$\overline{\lim_{\substack{n \to \infty \\ t \to 0}}} \; B(x_n, t_n) = \int_0^{\pi} \frac{\sin u}{u} \, du$$ and the Gibbs phenomenon occurs at $t = 0$.

On the other hand let f be any integrable function, periodic with period 2π. Let $s_n(x)$ be the nth partial sum of the Fourier series of $f(x)$. Then the following results are known:

Theorem 3.16:

Let $A = (a_{nk})$ be a summation method and let $\sigma_n(x) = \sum_{k=0}^{\infty} a_{nk} \, s_k(x)$.

Let $$\sum_{k=0}^{\infty} a_{nk} \frac{\sin(k+\frac{1}{2})x}{2 \sin \frac{1}{2} x} > 0$$ for all n and x.

Then $\{\sigma_n(x)\}$ does not exhibit the Gibbs phenomenon at any point.

This theorem appears to be well - known. Prasad and Siddiqi [16] for example, take it for granted in their paper.

Theorem 3.17 (Prasad - Siddiqi [16]):

Let $\sigma_n(x)$ be the nth Nörlund mean of $\{s_n(x)\}$. Let $p_n > 0$ and monotonic non-decreasing. Then $\{\sigma_n(x)\}$ does not exhibit the Gibbs phenomenon at any point. (The proof of this theorem depends on the preceding theorem).

Theorem 3.18 (Kuttner [10]):

Let $A = (a_{nk})$ be a k - method and let $\sigma_n(x) = \sum_{k=0}^{\infty} a_{nk}\, s_k(x)$. Let $\sum_{k=0}^{\infty} a_{nk} \dfrac{\sin(k + \frac{1}{2})x}{2 \sin\frac{1}{2} x}$ be bounded below for all n and x. Then $\{\sigma_n(x)\}$ does not exhibit the Gibbs phenomenon at any point.

The converse of this theorem is also true.

Another type of result may be obtained by placing restrictions on the function concerned. In the following f is an integrable function, periodic with period 2π, and $s_n(x)$ is the nth partial sum of the Fourier series of $f(x)$.

Theorem 3.19:

Let f be of bounded variation in an interval I and let I' be an interval interior to I. Then, for $\xi \in I'$,

$$\varlimsup_{\substack{n \to \infty \\ x \to \xi}} s_n(x) = \frac{f(\xi + 0) + f(\xi - 0)}{2}$$

$$+ \frac{f(\xi + 0) - f(\xi - 0)}{} \int_0^{\pi} \frac{\sin u}{u}\, du$$

$$> \frac{f(\xi + 0) + f(\xi - 0)}{2},$$

if f is discontinuous at $x = \xi$.

Hence $\{s_n(x)\}$ exhibits the Gibbs phenomenon at each point of discontinuity of f in I' and only there.

This result and the one following are well - known (see, eg., Zygmund [22]).

Theorem 3.20:

Let the modulus of continuity of f be $o(\frac{1}{|\log\delta|})$ in an interval I. Then $\{s_n(x)\}$ converges uniformly in every interval I' interior to I and hence, for $\xi \in I'$, $\{s_n(x)\}$ does not exhibit the Gibbs phenomenon at $x = \xi$.

Theorem 3.21: (Izumi - Satô [8]):

Let $$\int_0^h \{f(x+u) - f(x-u)\}\, du = o(\frac{h}{\log h})$$ uniformly in x as $h \to 0$. Then $\{s_n(x)\}$ does not exhibit the Gibbs phenomenon at any point.

Theorem 3.22 (Izumi - Satô [8]):

Let $$\int_0^h \{f(x+u) - f(x-u)\}\, du = o(h)$$ uniformly in x as $h \to 0$. Let $\sigma_n^r(x)$ be the nth Cesàro mean of order r of $\{s_n(x)\}$. Then, for $r > 0$, $\{\sigma_n^r(x)\}$ does not exhibit the Gibbs phenomenon at $x = 0$.

This will give the reader an idea of the type of results which have been proved concerning the Gibbs phenomenon. The bibliography shows that much of this work has been done recently.

For each new limiting process which is devised, the question of whether the transform of the sequence

$$\sum_{k=1}^{n} \frac{\sin k x}{k} \quad \text{exhibits the Gibbs phenomenon}$$

will naturally arise.

The generality of the weight function in the Hausdorff summation method leads to the question of finding necessary and sufficient conditions that the method preserve the Gibbs phenomenon of the same sequence. Livingston [38] and Newman [15], for example, have partially answered this question.

The problem of finding general classes of functions whose Fourier series exhibit the Gibbs phenomenon is one which has been attacked recently by several Japanese mathematicians. They have extended some known results to include functions having discontinuities of the second kind.

Questions along these lines and many other questions concerning the Gibbs phenomenon are far from being completely answered.

BIBLIOGRAPHY

[1] Barlaz, J. On Some Triangular Summation Methods. American Journal of Mathematics, Vol. LXIX, No. 1 (1947), pp. 139-152.

[2] Cheng, M. The Gibbs Phenomenon and Bochner's Summation Method I. Duke Mathematical Journal, 17, (1950), pp. 83-90.

[3] Cramér, H. Etudes sur la Sommation des Séries de Fourier, AFM, 13, No. 20, (1919), pp. 1-21.

[4] Graves, L. M. The Theory of Functions of Real Variables. McGraw-Hill, 1946.

[5] Hardy, G. Divergent Series. Clarenden Press, Oxford, 1949.

[6] Harsiladze, F. I. Gibbs' Phenomenon in the Summation of Fourier Series by the Methods of Bernštein - Rogosinski. Dokl. Akad. Nauk., SSSR. (NS), 101 (1955), pp. 425-428.

[7] Hobson, G. H. The Theory of Functions of a Real Variable, Vol. I. Cambridge University Press, 1927.

[8] Izumi, S. and Satô, M. Fourier Series XVI, The Gibbs Phenomenon of Partial Sums and Cesàro Means of Fourier Series 1. Proc. Japan Acad., Vol. 33. No. 6 (1957), pp. 284-288.

[9] Knopp, K. Theory and Applications of Infinite Series. Blackie and Son, London and Glascow, 1948.

[10] Kuttner, B. Note on the Gibbs Phenomenon. J. London Math. Soc. 20, (1943) pp. 136-139.

[11] Kuttner, B. On the Gibbs Phenomenon for Riesz Means. J. London Math. Soc. 19, (1944) pp. 153-161.

[12] Lee, C. On the Gibbs Phenomenon for the Riemann Summation (R,1) of Fourier Series. Acta Math. Sinica, 6 (1956), pp. 418-425.

[13] Lorch, L. The Gibbs Phenomenon for Borel Means. Proc. A.M.S., Vol. 8, No. 1, (1957) pp. 81-84.

[14] Miracle, C. L. The Gibbs Phenomenon for Taylor Means and for $[F, d_n]$ Means. Can. J. Math., Vol. 12 (1960) pp. 660-673.

[15] Newman, D. J. Gibbs Phenomenon for Hausdorff Means. Forthcoming in PAC. J. Math.

[16] Prasad, B. N. and Siddiqi, J. A. On the Gibbs Phenomenon for the Norlund Means. Indian. J. Math. Vol. 1 (1959), pp. 21-27.

[17] Rogosinski, W. W. Fourier Series. Chelsea Publishing Company, New York.

[18] Sledd, W. T. Gibbs Phenomenon and Lebesgue Constants for Regular Sonnenschein Matrices. Forthcoming.

[19] Szasz, O. On the Gibbs Phenomenon for Euler Means. Acta Univ. Szeged., Vol. 12, part B (1950), pp. 107-111.

[20] Szasz, O. Gibbs' Phenomenon for Hausdorff Means. Trans. Amer. Math. Soc., Vol. 69 (1950), pp. 440-456.

[21] Zalcwasser, Z. Sur le Phénomène de Gibbs das la Théorie des Séries de Fourier des Fonctions Continues. Fundamenta Mathematica, 12 (1928), pp. 126-151.

[22] Zygmund, A. Trigonometric Series, Vol. I. Cambridge Univ. Press, 1959.

[23] Cheng, M. The Gibbs Phenomenon and Bochner's Summation Method, I. Duke Math. J. 17 (1950), pp. 83-90.

[24] Cheng, M. The Gibbs Phenomenon and Bochner's Summation Method, II. Duke Math. J. 17 (1950), pp. 477-490.

[25] Gibbs, W. Fourier Series, Nature 59 (1899) pp. 606.

[26] Gronwall, T. H. Zur Gibbsschen Erscheinung. Ann. of Math. 31 (1930), pp. 233-240.

[27] Hardy, G. H. and Rogosinski, W. W. Notes on Fourier Series II, on the Gibbs Phenomenon. J. London Math. Soc. 18 (1943), pp. 83-87.

[28] Hardy, G. H. and Rogosinski, W. W. Fourier Series, Cambridge Univ. Press, 1944.

[29] Ishiguro, K. Fourier Series XI, Gibbs' Phenomenon. Kodai Math. Sem. Rep. 8 (1956), pp. 181-188.

[30] Ishiguro, K. Correction to the paper "Fourier Series XI, Gibbs' Phenomenon". Kodai Math. Sem. Rep. 9 (1957), pp. 191-192.

[31] Ishiguro, K. Fourier Series XV, Gibbs' Phenomenon. Proc. Japan Acad. 33 (1957), pp. 119-123.

[32] Ishiguro, K. On the Gibbs Phenomenon. Journal of the Faculty of Science, Hokkaido University, Series I, Vol. XV, No. 1, 2 (1960), pp. 145-171.

[33] Ishiguro, K. Zur Gibbsschen Erscheinung fur das Kreisverfahren. Math. Zeitschr. 76 (1961), pp. 288-294.

[34] Izumi, S. and Satô, M. Fourier Series X: Rogosinski's Lemma. Kodai Math. Sem. Rep. 8 (1956), pp. 164-180.

[35] Izumi, S. and Satô, M. The Gibbs Phenomenon of Partial Sums and Cesaro Means of Fourier Series, 2. Proc. Japan Acad. 33 (1957), pp. 290-292.

[36] Kuttner, B. Further Note on the Gibbs Phenomenon. J. London Math. Soc. 22 (1947), pp. 295-298.

[37] Lee, C. On the Gibbs Phenomenon for Riemann Summation (R,k) of Generalized Fourier Series. Acta Math. Sinica (1959), pp. 28-36.

[38] Livingston, A. E. Some Hausdorff Means which Exhibit the Gibbs Phenomenon. Pac. J. Math. 3 (1953), pp. 407-415.

[39] Szasz, O. On Some Trigonometrical Summability Methods and Gibbs' Phenomenon. Trans. Amer. Math. Soc. 54 (1943), pp. 483-497.

[40] Szasz, O. On the Partial Sums of Fourier Series at Points of Discontinuity. Trans. Amer. Math. Soc. 53 (1943), pp. 440-453.

[41] Szasz, O. The Generalized Jump of a Function and Gibbs Phenomenon. Duke Math. J. 11 (1944), pp. 823-833.

[42] Szasz, O. On the Gibbs Phenomenon for a Class of Linear Transforms. Acad. Serbe Sci, Publ. Inst. Math. 4 (1952), pp. 135-144.

[43] Walmsley, C. Gibbs Phenomenon for Cesàro and Hölder Summation of Generalized Fourier Series. J. London Math. Soc. 28 (1953), pp. 148-156.